

An Improved Combes-Thomas Estimate of Magnetic Schrödinger Operators

Zhongwei Shen*

Department of Mathematics and Statistics
Auburn University
Auburn, AL 36849
USA

July 17, 2012

Abstract

In the present paper, we prove an improved Combes-Thomas estimate, i.e., the Combes-Thomas estimate in trace-class norms, for magnetic Schrödinger operators under general assumptions. In particular, we allow unbounded potentials. We also show that for any function in the Schwartz space on the reals the operator kernel decays, in trace-class norms, faster than any polynomial.

Keywords: magnetic Schrödinger operator, Combes-Thomas estimate, trace ideal estimate, operator kernel estimate.

2010 Mathematics Subject Classification: Primary 81Q10, 47F05; Secondary 35P05.

Contents

1	Introduction	2
2	Standing Notations	4
3	Semigroup and Trace Ideal Estimates	5
4	The Combes-Thomas Estimate in Trace Ideals	8
5	The Operator Kernel Estimate in Trace Ideals	16
A	Sectorial Form and m-Sectorial Operator	20
B	Justification of (37)	21

*Email: zzs0004@auburn.edu

1 Introduction

This paper is concerned with the so called Combes-Thomas estimate of the following Schrödinger operator with magnetic field

$$H_\Lambda(A, V) = \frac{1}{2}(-i\nabla - A(x))^2 + V(x) \quad \text{on } \Lambda, \quad (1)$$

where $i = \sqrt{-1}$ is the imaginary unit, $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_d})$ is the gradient, A is the vector potential giving rise to the magnetic field $\nabla \times A$, V is the electric potential and $\Lambda \subset \mathbb{R}^d$ is the configuration space with dimension d . This operator is used to characterize a spinless particle subject to a scalar potential and a magnetic field in non-relativistic quantum physics [20, 21, 40].

As it is known, the Combes-Thomas estimate plays an important role in the theory of Schrödinger operators, magnetic Schrödinger operators, classical wave operators, etc. in random media. It was invented by Combes and Thomas [9] to study the asymptotic behavior of eigenfunctions for multi-particle Schrödinger operators. Later, Fröhlich and Spencer [15] used it to study the localization for the multidimensional discrete Anderson model. Meanwhile, the Combes-Thomas estimate, as well as Wegner estimate [42] and Lifshitz tail [31], became important ingredients in multiscale analysis. Specifically, the initial scale estimate in multiscale analysis for localization near the bottom of the spectrum is successful because of the Combes-Thomas estimate. See [1, 4, 7, 13, 14, 17, 18, 19, 25, 27, 28, 29, 35, 38] and references therein for further applications. Moreover, a stronger version of the Combes-Thomas estimate, i.e., the estimate in trace-class norms, is very useful. In [8] and [26], such estimates have been applied to study the regularity of the integrated density of states, a concept of great physical significance [32]. See [3, 30] for other applications.

Since the pioneering work of Combes and Thomas [9], the Combes-Thomas estimate in operator norm has been well studied (see [1, 13, 14, 28, 35, 38] and reference therein). We point out the work of Germinet and Klein [18]. They proved a Combes-Thomas estimate, in operator norm, with explicit bound of general Schrödinger operators including Schrödinger operator, magnetic Schrödinger operator, acoustic operator, Maxwell operator and so on. However, most existing results about the Combes-Thomas estimate in trace-class norms were proven, more or less, under additional assumptions. For instance, Barbaroux, Combes and Hislop proved in [3] the estimate under the assumption of some sort of analyticity. Klopp's result, obtained in [30], for Schrödinger operators without magnetic fields was proven under the assumption of the boundedness of the potential. Results about the Combes-Thomas estimate in trace-class norms under general assumptions are unknown so far.

The main goal of the current paper is to obtain the Combes-Thomas estimate of (1) and the associated operator kernel estimate in trace-class norms under general assumptions, which allow the potential to be unbounded. We first prove an improved Combes-Thomas estimate, i.e., the Combes-Thomas estimate in trace-class norms, for the magnetic Schrödinger operator (1) under general assumptions. Based on the improved Combes-Thomas estimate, we also show that for any function in the Schwartz space on the reals the operator kernel decays, in trace-class norms, faster than any polynomial.

To be more specific, we assume that the magnetic vector potential $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$ is \mathbb{R}^d -valued, the electric potential $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ is real-valued and the dimension $d \geq 2$. The notations $\mathcal{H}_{loc}(\mathbb{R}^d)$ and $\mathcal{K}_{\pm}(\mathbb{R}^d)$ for spaces are explained in Section 2. Let $\Lambda \subset \mathbb{R}^d$ be an open set. We assume that Λ is bounded with sufficiently smooth boundary if it is not the whole space. The self-adjoint realization of $H_{\Lambda}(A, V)$ on $L^2(\Lambda)$ is still denoted by $H_{\Lambda}(A, V)$. If $\Lambda \neq \mathbb{R}^d$, then $H_{\Lambda}(A, V)$ is nothing but the localized operator with homogeneous Dirichlet boundary on $\partial\Lambda$. These self-adjoint operators are constructed via sesquilinear forms. In Section 3, we will recall the constructions done in [5].

Our first purpose is to study the Combes-Thomas estimate in trace class norms, i.e., the trace ideal estimate of the operators

$$\chi_{\beta}(H_{\Lambda}(A, V) - z)^{-n}\chi_{\gamma}, \quad \beta, \gamma \in \mathbb{R}^d,$$

where χ_{β} is the characteristic function of the unit cube centered at $\beta \in \mathbb{R}^d$ and $z \in \rho(H_{\Lambda}(A, V))$, the resolvent set of $H_{\Lambda}(A, V)$. More precisely, we want to obtain the exponential decay of $\|\chi_{\beta}(H_{\Lambda}(A, V) - z)^{-n}\chi_{\gamma}\|_{\mathcal{J}_p}$ in terms of $|\beta - \gamma|$ for suitable n and p , where $\|\cdot\|_{\mathcal{J}_p}$ is the p -th von Neumann-Schatten norm reviewed in Section 2. Following the definition in [18], the family of operators $\{\chi_{\beta}(H_{\Lambda}(A, V) - z)^{-n}\chi_{\gamma}\}_{\beta, \gamma \in \mathbb{R}^d}$ is also called the operator kernel of the bounded operator $(H_{\Lambda}(A, V) - z)^{-n}$. In general, if f is a bounded Borel function on $\sigma(H_{\Lambda}(A, V))$, the spectrum of $H_{\Lambda}(A, V)$, then the family $\{\chi_{\beta}f(H_{\Lambda}(A, V))\chi_{\gamma}\}_{\beta, \gamma \in \mathbb{R}^d}$ is called the operator kernel of the bounded linear operator $f(H_{\Lambda}(A, V))$. Our first main result regarding the Combes-Thomas estimate is roughly stated as follows (see Theorem 4.5 and Theorem 4.6 for details).

Theorem 1.1. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Suppose $p > \frac{d}{2n}$ with $n \in \mathbb{N}$ and $n \geq 1$. For any $z \in \rho(H_{\Lambda}(A, V))$, the resolvent set of $H_{\Lambda}(A, V)$, there exist constants $C = C(p, z, n) > 0$ and $a_0 = a_0(z) > 0$ such that*

$$\|\chi_{\beta}(H_{\Lambda}(A, V) - z)^{-n}\chi_{\gamma}\|_{\mathcal{J}_p} \leq Ce^{-a_0|\beta - \gamma|}, \quad \forall \beta, \gamma \in \mathbb{R}^d. \quad (2)$$

In this paper, we also study operator kernel estimate in trace-class norms. That is, we prove the polynomial decay of the operators

$$\chi_{\beta}f(H_{\Lambda}(A, V))\chi_{\gamma}, \quad \beta, \gamma \in \mathbb{R}^d$$

in trace-class norms in terms of $|\beta - \gamma|$, where f belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ reviewed in Section 2. The main result related to operator kernel estimate is roughly stated as follows (see Theorem 5.2 for details).

Theorem 1.2. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Suppose $p > \frac{d}{2}$. Then, for any $f \in \mathcal{S}(\mathbb{R})$ and any $k \in \mathbb{Z}$ with $k \geq 1$, there exists a constant $C = C(p, k, f) > 0$ such that*

$$\|\chi_{\beta}f(H_{\Lambda}(A, V))\chi_{\gamma}\|_{\mathcal{J}_p} \leq C|\beta - \gamma|^{-k}, \quad \forall \beta, \gamma \in \mathbb{R}^d. \quad (3)$$

Estimates like (3), with A being \mathbb{Z}^d -period, V being bounded and f being a smooth function with compact support, have been used, as a technical tool, to study the regularity of integrated density of states. For instance, Combes, Hislop and Klopp [8, Eq.(2.30)] utilize

the polynomial decay of any order to prove the convergence of some series, which leads to an expected estimate. It should be pointed out that Germinet and Klein proved in [18] for slowly decreasing smooth functions (see Appendix C for the definition) the operator kernels for general Schrödinger operators decay, in the operator norm, faster than any polynomial. Their result was then used as a crucial ingredient in their following paper [19].

The rest of the paper is organized as follows. In Section 2, we collect the notations used in this paper. In Section 3, we study trace ideal estimates of operators of the form $gf(H_\Lambda(A, V))$ for suitable f and g . Such estimates, with g being characteristic functions of unit cubes and f being integer powers of the resolvent of $H_\Lambda(A, V)$, are used as technical tools in the proof of (2). Section 4 is devoted to the study of the Combes-Thomas estimate in trace-class norms. That is, we prove Theorem 1.1. In Section 5, we study the operator kernel estimate in trace-class norms and prove Theorem 1.2.

2 Standing Notations

In this section, we collect the notations which will be used in the sequel.

The configuration space Λ is an open set of \mathbb{R}^d . We assume that Λ is bounded with sufficiently smooth boundary unless it is the whole space. We also assume that the dimension $d \geq 2$ since, by gauge transform, vector potentials in one spatial dimension are of no physical interest.

We denote by χ_β the characteristic function of the unit cube centered at $\beta \in \mathbb{R}^d$. If the configuration space in question is $\Lambda (\neq \mathbb{R}^d)$, then χ_β should be understood as $\chi_\beta \chi_\Lambda$, where χ_Λ is the characteristic function of Λ . Generally speaking, if a function is defined on Λ , then we consider it as a function defined on \mathbb{R}^d by zero extension on $\mathbb{R}^d \setminus \Lambda$.

The Banach space of p -th Lebesgue integrable functions on Λ is

$$L^p(\Lambda) = \{ \phi \text{ measurable on } \Lambda \mid \|\phi\|_p < \infty \},$$

where $\|\phi\|_p = (\int_\Lambda |\phi(x)|^p dx)^{\frac{1}{p}}$ if $p \in [1, \infty)$ and $\|\phi\|_\infty = \text{ess sup}_{x \in \Lambda} |\phi(x)|$. When $p = 2$, $L^2(\Lambda)$ is a Hilbert space with inner product

$$\langle \phi, \psi \rangle = \int_\Lambda \bar{\phi}(x) \psi(x) dx.$$

Moreover, $\|\phi\|_2 = \sqrt{\langle \phi, \phi \rangle}$. As a convention, we simply write $\|\cdot\|_2$ as $\|\cdot\|$.

If $L : L^p(\Lambda) \rightarrow L^q(\Lambda)$ is a bounded linear operator, the operator norm is defined by

$$\|L\|_{p,q} := \sup_{\|\phi\|_p=1} \|L\phi\|_q.$$

If $p = q = 2$, we simply write $\|\cdot\|_{2,2}$ as $\|\cdot\|$.

Although we use the same notation $\|\cdot\|$ for both the norm of a function in $L^2(\Lambda)$ and the norm of an operator on $L^2(\Lambda)$, it should not give rise to any confusion. Similarly, we do not distinguish the notations for norms corresponding to different configuration spaces.

For any $p \in [1, \infty)$, the Banach space \mathcal{J}_p (also an operator ideal) is defined by

$$\mathcal{J}_p = \{ C : L^2(\Lambda) \rightarrow L^2(\Lambda) \text{ linear and bounded} \mid \|C\|_{\mathcal{J}_p} < \infty \},$$

where $\|C\|_{\mathcal{J}_p} = (\text{Tr}|C|^p)^{\frac{1}{p}} < \infty$ is the p -th von Neumann-Schatten norm of C . See [33, 36] for more details. We here single out the space \mathcal{J}_2 (also called the space of Hilbert-Schmidt operators) for the following important property (see [34, Theorem VI.23]): a bounded linear operator K on $L^2(\Lambda)$ belongs to \mathcal{J}_2 if and only if it is an integral operator with some integral kernel $k(x, y)$ being in $L^2(\Lambda \times \Lambda)$. In this case, $\|K\|_{\mathcal{J}_2} = (\int_{\Lambda \times \Lambda} |k(x, y)|^2 dx dy)^{\frac{1}{2}}$. We will use this property in Section 3.

Let $g(x) = -\ln|x|$ if $d = 2$ and $g(x) = |x|^{2-d}$ if $d \geq 3$. We say a function $V \in \mathcal{K}(\mathbb{R}^d)$, the Kato class, if

$$\lim_{\epsilon \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \epsilon} g(x-y) |V(y)| dy = 0.$$

A function V is said to be in the local Kato class $\mathcal{K}_{loc}(\mathbb{R}^d)$ if $V\chi_K \in \mathcal{K}(\mathbb{R}^d)$ for all compact set $K \subset \mathbb{R}^d$, where χ_K is the characteristic function of K . We refer to [39] for equivalent definitions from the viewpoint of probability theory.

Let V defined on \mathbb{R}^d be real-valued. We say that V is Kato decomposable, in symbols $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$, if the positive part V_+ is in $\mathcal{K}_{loc}(\mathbb{R}^d)$ and the negative part V_- is in $\mathcal{K}(\mathbb{R}^d)$.

A \mathbb{C}^d -valued function A is said to be in the class $\mathcal{H}(\mathbb{R}^d)$ if its squared norm $A \cdot A$ and its divergence $\nabla \cdot A$, considered as a distribution on $C_0^\infty(\mathbb{R}^d)$, are both in the Kato class $\mathcal{K}(\mathbb{R}^d)$. It is said to be in the class $\mathcal{H}_{loc}(\mathbb{R}^d)$ if both $A \cdot A$ and $\nabla \cdot A$ are in the local Kato class $\mathcal{K}_{loc}(\mathbb{R}^d)$. We refer the reader to [2, 5, 6, 10] for further remarks about these spaces.

The Schwartz space $\mathcal{S}(\mathbb{R})$ consists of those $C^\infty(\mathbb{R})$ functions which, together with all their derivatives, vanish at infinity faster than any power of $|x|$. More precisely, for any $N \in \mathbb{Z}$, $N \geq 0$ and any $r \in \mathbb{Z}$, $r \geq 0$, we define for $f \in C^\infty(\mathbb{R})$

$$\|f\|_{N,r} = \sup_{x \in \mathbb{R}} (1 + |x|)^N |f^{(r)}(x)|,$$

then

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) \mid \|f\|_{N,r} < \infty \text{ for all } N, r\}.$$

See Folland [16] for more discussions about the Schwartz space.

3 Semigroup and Trace Ideal Estimates

In this section, as a preparation for proving Theorem 1.1 and Theorem 1.2, we study estimates of operators of the form $gf(H_\Lambda(A, V))$ in trace-class norms for suitable f and g .

The self-adjoint realization of $H_\Lambda(A, V)$ on $L^2(\Lambda)$, still denoted by $H_\Lambda(A, V)$, is defined via sesquilinear forms as follows (see [5]): the sesquilinear form

$$h_\Lambda^{A, V_+} : C_0^\infty(\Lambda) \times C_0^\infty(\Lambda) \rightarrow \mathbb{C},$$

$$(\psi, \phi) \mapsto h_\Lambda^{A, V_+}(\psi, \phi) := \langle \sqrt{V_+} \psi, \sqrt{V_+} \phi \rangle + \frac{1}{2} \sum_{j=1}^d \langle (-i\partial_j - A_j) \psi, (-i\partial_j - A_j) \phi \rangle$$

is densely defined in $L^2(\Lambda)$, nonnegative and closable, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(\Lambda)$. Its closure is still denoted by h_Λ^{A, V_+} with form domain $\mathcal{Q}(h_\Lambda^{A, V_+})$, which is

the completion of $C_0^\infty(\Lambda)$ with respect to the norm

$$\|\phi\|_{h_\Lambda^{A,V_+}} = \sqrt{\|\phi\|^2 + h_\Lambda^{A,V_+}(\phi, \phi)},$$

where $\|\cdot\| = \|\cdot\|_2$ is the norm on $L^2(\Lambda)$ associated with $\langle \cdot, \cdot \rangle$ as mentioned in Section 2. We denote by $H_\Lambda(A, V_+)$ the associated self-adjoint operator. Since $V_- \in \mathcal{K}(\mathbb{R}^d)$ is infinitesimally form-bounded with respect to $H_\Lambda(A, 0) (\leq H_\Lambda(A, V_+))$, i.e., there exist $\Theta_1 \in (0, 1)$ (can be taken to be arbitrarily small) and $\Theta_2 \geq 0$ depending on Θ_1 so that

$$\langle \phi, V_- \phi \rangle \leq \Theta_1 h_\Lambda^{A,0}(\phi, \phi) + \Theta_2 \|\phi\|^2, \quad \phi \in \mathcal{Q}(h_\Lambda^{A,0}), \quad (4)$$

KLMN theorem (see [34, Theorem X.17]) yields that, with $\mathcal{Q}(h_\Lambda^{A,V}) = \mathcal{Q}(h_\Lambda^{A,V_+})$, the sesquilinear form

$$\begin{aligned} h_\Lambda^{A,V} : \mathcal{Q}(h_\Lambda^{A,V}) \times \mathcal{Q}(h_\Lambda^{A,V}) &\rightarrow \mathbb{C}, \\ (\psi, \phi) &\mapsto h_\Lambda^{A,V}(\psi, \phi) := h_\Lambda^{A,V_+}(\psi, \phi) - \langle \sqrt{V_-} \psi, \sqrt{V_-} \phi \rangle \end{aligned} \quad (5)$$

is closed and bounded from below and has $C_0^\infty(\Lambda)$ as a form core. The associated semi-bounded self-adjoint operator is denoted by $H_\Lambda(A, V)$.

The main result of this section is stated as follows. Let

$$E_0 = \text{the infimum of the } L^2(\mathbb{R}^d)\text{-spectrum of } H_{\mathbb{R}^d}(0, V). \quad (6)$$

Theorem 3.1. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Suppose $p \geq 2$. Let f be a Borel function satisfying*

$$|f(\lambda)| \leq C(1 + |\lambda|)^{-\alpha}, \quad \lambda \in \sigma(H_\Lambda(A, V)), \quad (7)$$

for $\alpha > \frac{d}{2p}$. Then $gf(H_\Lambda(A, V))$ is in \mathcal{J}_p with

$$\|gf(H_\Lambda(A, V))\|_{\mathcal{J}_p} \leq C_{\alpha,p,\lambda_0} \|g\|_p \|(H_\Lambda(A, V) - \lambda_0)^\alpha f(H_\Lambda(A, V))\|$$

whenever $g \in L^p(\Lambda)$, where $\lambda_0 < E_0$ and $C_{\alpha,p,\lambda_0} > 0$ depends only on α, p and λ_0 .

To prove the above theorem, we first present some lemmas. We begin with the celebrated Feynman-Kac-Itô formula proven by Broderix, Hundertmark and Leschke (See [24, 37, 39] and references therein for earlier versions).

Lemma 3.2 ([5]). *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. For any $\phi \in L^2(\Lambda)$ and $t \geq 0$, there holds*

$$(e^{-tH_\Lambda(A,V)} \phi)(x) = \mathbb{E}_x \{ e^{-S_t^\omega(A,V)} \Xi_{\Lambda,t}(\omega) \phi(\omega(t)) \} \quad \text{for a.e. } x \in \Lambda, \quad (8)$$

where

$$S_t^\omega(A, V) = i \int_0^t A(\omega(s)) d\omega(s) + \frac{i}{2} \int_0^t (\nabla \cdot A)(\omega(s)) ds + \int_0^t V(\omega(s)) ds,$$

$\mathbb{E}_x\{\cdot\}$ denotes the expectation for the Brownian motion starting at x and $\Xi_{\Lambda,t}$ is the characteristic function of the set $\{\omega | \omega(s) \in \Lambda \text{ for all } s \in [0, t]\}$.

As consequences of (8), we get the so called diamagnetic inequality

$$|e^{-tH_\Lambda(A,V)}\phi| \leq e^{-tH_\Lambda(0,V)}|\phi|, \quad t \geq 0, \quad (9)$$

the monotonicity of semigroup for vanishing magnetic field in the sense that for $\Lambda \subset \Lambda'$

$$e^{-tH_\Lambda(0,V)}\chi_\Lambda\phi \leq e^{-tH_{\Lambda'}(0,V)}\phi, \quad \phi \geq 0, \quad t \geq 0$$

and then the L^p -smoothing of semigroups: for $1 \leq p \leq q \leq \infty$, there exist constant $C > 0$ and E such that

$$\|e^{-tH_\Lambda(A,V)}\|_{p,q} \leq \|e^{-tH_\Lambda(0,V)}\|_{p,q} \leq \|e^{-tH_{\mathbb{R}^d}(0,V)}\|_{p,q} \leq Ct^{-\gamma}e^{Et}, \quad (10)$$

where $\gamma = \frac{d}{2}(\frac{1}{p} - \frac{1}{q})$. We remark that E can be chosen such that $-E < E_0$ (See [5, 35]).

We extend [35, Theorem B.2.1] to the magnetic case.

Lemma 3.3. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Let $\alpha > 0$ and $1 \leq p \leq q \leq \infty$ satisfy*

$$\frac{1}{p} - \frac{1}{q} < \frac{2\alpha}{d}. \quad (11)$$

Then $(H_\Lambda(A, V) - z)^{-\alpha}$ is bounded from $L^p(\Lambda)$ to $L^q(\Lambda)$ whenever the real part $\Re z < E_0$.

Proof. It follows from the formula

$$(H_\Lambda(A, V) - z)^{-\alpha} = c_\alpha \int_0^\infty e^{-tH_\Lambda(A,V)} e^{tz} t^{\alpha-1} dt \quad (12)$$

and (10), where the assumption (11) is applied to insure the convergence of the integral in (12). \square

As a consequence of Lemma 3.3, we have

Lemma 3.4. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Let $\alpha > 0$ and $1 \leq p \leq 2 \leq q \leq \infty$ satisfy (11). For any Borel function f satisfying (7), the operator $f(H_\Lambda(A, V))$ is bounded from $L^p(\Lambda)$ to $L^q(\Lambda)$ with*

$$\|f(H_\Lambda(A, V))\|_{p,q} \leq C_{p,q,\alpha,\lambda_0} \|(H_\Lambda(A, V) - \lambda_0)^\alpha f(H_\Lambda(A, V))\|, \quad (13)$$

where $\lambda_0 < E_0$ and $C_{p,q,\alpha,\lambda_0} > 0$ depends only on p, q, α and λ_0 .

Proof. It follows from the arguments in [35, Theorem B.2.3]. \square

We next discuss the trace ideal estimate of operators of the form $gf(H_\Lambda(A, V))$ for suitable f and g . We start with recalling a result of Dunford and Pettis (See [10, 35, 41] for abstract versions).

Lemma 3.5. *Let (M, μ) be a separable measurable space. If L is a bounded linear operator from $L^p(M)$ to $L^\infty(M)$ with $1 \leq p < \infty$, then there is a measurable function $k(\cdot, \cdot)$ on $M \times M$ such that L is an integral operator with integral kernel $k(\cdot, \cdot)$ and*

$$\sup_{x \in M} \left(\int_M |k(x, y)|^{p'} d\mu(y) \right)^{\frac{1}{p'}} = \|L\|_{p,\infty} < \infty,$$

where $p' = \frac{p}{p-1}$ is the conjugate exponent of p .

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By complex interpolation (see [36, Theorem 2.9]), it suffices to prove the result in the case $p = 2$, which we show now. For $p = 2$ and $q = \infty$, we have $\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) = \frac{d}{4} < \alpha$ by assumption, i.e., (11) is satisfied, and thus, Lemma 3.4 implies that $f(H_\Lambda(A, V))$ is bounded from $L^2(\Lambda)$ to $L^\infty(\Lambda)$. By Lemma 3.5, $f(H_\Lambda(A, V))$ is an integral operator with kernel $k_\Lambda^{A,V}(x, y)$ satisfying

$$\sup_{x \in \Lambda} \int_{\Lambda} |k_\Lambda^{A,V}(x, y)|^2 dy = \|f(H_\Lambda(A, V))\|_{2,\infty}^2 < \infty.$$

Thus, $gf(H_\Lambda(A, V))$ is an integral operator on $L^2(\Lambda)$ with kernel $g(x)k_\Lambda^{A,V}(x, y)$. Moreover,

$$\iint_{\Lambda \times \Lambda} |g(x)k_\Lambda^{A,V}(x, y)|^2 dx dy \leq \|g\|_2^2 \sup_{x \in \Lambda} \int_{\Lambda} |k_\Lambda^{A,V}(x, y)|^2 dy = \|g\|_2^2 \|f(H_\Lambda(A, V))\|_{2,\infty}^2,$$

which implies that $gf(H_\Lambda(A, V))$ is a Hilbert-Schmidt operator as mentioned in Section 2, i.e., in \mathcal{J}_2 , with \mathcal{J}_2 -norm bounded by $\|g\|_2 \|f(H_\Lambda(A, V))\|_{2,\infty}$. The expected bound is given by (13). This completes the proof. \square

We remark that results obtained in this section are well-known for Schrödinger operators without magnetic fields. See [2, 35] and references therein. It should be pointed out that the result of Theorem 3.1 in the case $H_{\mathbb{R}^d}(0, V)$ was proven in [35, Theorem B.9.3] for any $p \geq 1$. To prove the result for $p \in [1, 2)$, it was first shown that $gf(H_{\mathbb{R}^d}(0, V)) \in \mathcal{J}_1$ for $g \in \ell^1(L^2(\mathbb{R}^d))$, the Birman-Solomjak space, then proceeded to complex interpolation. The proof relies on the translation invariance of the free Laplacian (see [35, Theorem B.9.2] and [36, Theorem 4.5] for instance), which, however, is not true for magnetic Schrödinger operators. This prevents us from obtaining the result for $p \in [1, 2)$.

4 The Combes-Thomas Estimate in Trace Ideals

In this section, we study the improved Combes-Thomas estimate, i.e., the trace ideal estimate of the operators

$$\chi_\beta(H_\Lambda(A, V) - z)^{-n} \chi_\gamma \quad \text{for } \beta, \gamma \in \mathbb{R}^d,$$

where χ_β is the characteristic function of the unit cube centered at β . More precisely, we want to obtain the exponential decay of $\|\chi_\beta(H_\Lambda(A, V) - z)^{-1} \chi_\gamma\|_{\mathcal{J}_p}$ in terms of $|\beta - \gamma|$. The main result is stated in Theorem 1.1. Since we also consider localized operators, χ_β should be understood as $\chi_\beta \chi_\Lambda$ if the operators is restricted to Λ as it is mentioned in Section 2, where χ_Λ is the characteristic function of the domain Λ . The basic tools we use here are sectorial form and m -sectorial operator reviewed in Appendix A. We also employ the classical argument of Combes and Thomas developed in [9].

First of all, we establish some results by applying the theory of sectorial form and m -sectorial operator. For this purpose, we first define auxiliary sesquilinear forms with associated operators formally given by

$$H_\Lambda^a(A, V) = e^{a \cdot x} H_\Lambda(A, V) e^{-a \cdot x}, \quad a \in \mathbb{R}^d, \quad (14)$$

where $e^{a \cdot x}$ and $e^{-a \cdot x}$ are multiplicative operators. Note that the operator $H_\Lambda^a(A, V)$ is not self-adjoint unless $a = 0$. First, we denote by $D_{A, \Lambda}$ the closure of $\frac{\sqrt{2}}{2}(-i\nabla - A)$ on $C_0^\infty(\Lambda)$, so $H_\Lambda(A, 0) = D_{A, \Lambda}^* D_{A, \Lambda}$. This can be seen by sesquilinear forms. Moreover, the domain of $D_{A, \Lambda}$, denoted by $\mathcal{D}(D_{A, \Lambda})$, is the form domain, denoted by $\mathcal{Q}(h_\Lambda^{A, 0})$, of the sesquilinear form associated with the lower bounded self-adjoint operator $H_\Lambda(A, 0)$. For $a \in \Lambda$, we define

$$D_{A, \Lambda}(a) = e^{a \cdot x} D_{A, \Lambda} e^{-a \cdot x} \quad \text{and} \quad D_{A, \Lambda}^*(a) = e^{a \cdot x} D_{A, \Lambda}^* e^{-a \cdot x}.$$

It's easy to see that

$$\begin{aligned} D_{A, \Lambda}(a) &= D_{A, \Lambda} + i \frac{\sqrt{2}}{2} a, \quad \text{on } \mathcal{D}(D_{A, \Lambda}), \\ D_{A, \Lambda}^*(a) &= D_{A, \Lambda}^* + i \frac{\sqrt{2}}{2} a, \quad \text{on } \mathcal{D}(D_{A, \Lambda}^*) \end{aligned} \tag{15}$$

and they are closed, densely defined operators. Note that $(D_{A, \Lambda}(a))^* \neq D_{A, \Lambda}^*(a)$. Next, we define the sesquilinear form $h_\Lambda^{A, 0}(a)$ on $\mathcal{D}(D_{A, \Lambda}) = \mathcal{Q}(h_\Lambda^{A, 0})$ by

$$h_\Lambda^{A, 0}(a)(\psi, \phi) = \langle (D_{A, \Lambda}^*(a))^* \psi, D_{A, \Lambda}(a) \phi \rangle. \tag{16}$$

Obviously, $h_\Lambda^{A, 0}(0) \equiv h_\Lambda^{A, 0}$. Finally, we define the sesquilinear form $h_\Lambda^{A, V}(a)$ on $\mathcal{Q}(h_\Lambda^{A, V+})$ by

$$h_\Lambda^{A, V}(a)(\psi, \phi) = h_\Lambda^{A, 0}(a)(\psi, \phi) + \langle \sqrt{V_+} \psi, \sqrt{V_+} \phi \rangle - \langle \sqrt{V_-} \psi, \sqrt{V_-} \phi \rangle. \tag{17}$$

For notational simplicity, we let

$$c_s = \frac{2\Theta_2 s}{1 - \Theta_1} + \frac{1}{2} \left(\frac{1}{s} + s \right), \quad s > 0 \tag{18}$$

where Θ_1, Θ_2 are given in (4). For $a_0 > 0$, let

$$\Xi_1(s) = \frac{2s}{1 - \Theta_1}, \quad \Xi_2(s, a_0) = c_s a_0^2. \tag{19}$$

We will write $\Xi_1(s)$ and $\Xi_2(s, a_0)$ as Ξ_1 and Ξ_2 , respectively, in the sequel.

We next prove several lemmas related to $H_\Lambda^a(A, V)$. Our first lemma is about the relation between $h_\Lambda^{A, V}(a)$ and $H_\Lambda^a(A, V)$.

Lemma 4.1. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. The sesquilinear form $h_\Lambda^{A, V}(a)$ defined in (17) is a closed sectorial form associated with the unique m -sectorial operator $H_\Lambda^a(A, V)$ given by (14).*

Proof. By (5), (15), (16) and (17), we have for any $\phi \in \mathcal{Q}(h_\Lambda^{A, V})$,

$$\begin{aligned} |h_\Lambda^{A, V}(a)(\phi, \phi) - h_\Lambda^{A, 0}(a)(\phi, \phi)| &= |h_\Lambda^{A, 0}(a)(\phi, \phi) - h_\Lambda^{A, 0}(\phi, \phi)| \\ &\leq \sqrt{2} |\Re \langle \phi, a \cdot D_{A, \Lambda} \phi \rangle| + \frac{1}{2} |a|^2 \|\phi\|^2 \end{aligned}$$

so that

$$|h_{\Lambda}^{A,V}(a)(\phi, \phi) - h_{\Lambda}^{A,V}(\phi, \phi)|^2 \leq 4|a|^2\|\phi\|^2\|D_{A,V}\phi\|^2 + \frac{1}{2}|a|^4\|\phi\|^4,$$

which implies that for any $s > 0$,

$$\begin{aligned} |h_{\Lambda}^{A,V}(a)(\phi, \phi) - h_{\Lambda}^{A,V}(\phi, \phi)| &\leq |a|\|\phi\| \left(4\|\phi\|^2\|D_{A,V}\phi\|^2 + \frac{1}{2}|a|^2\|\phi\|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2s}|a|^2\|\phi\|^2 + \frac{s}{2} \left(4\|\phi\|^2\|D_{A,V}\phi\|^2 + \frac{1}{2}|a|^2\|\phi\|^2 \right) \\ &= 2sh_{\Lambda}^{A,0}(\phi, \phi) + \left(\frac{1}{2s} + \frac{s}{2} \right) |a|^2\|\phi\|^2, \end{aligned} \quad (20)$$

since $h_{\Lambda}^{A,0}(\phi, \phi) = \|D_{A,V}\phi\|^2$. Thanks to (4) and (5),

$$h_{\Lambda}^{A,V} \geq (1 - \Theta_1)h_{\Lambda}^{A,0} - \Theta_2 \quad \text{on} \quad \mathcal{Q}(h_{\Lambda}^{A,V}) \subset \mathcal{Q}(h_{\Lambda}^{A,0}).$$

This, together with (20), implies that

$$|h_{\Lambda}^{A,V}(a)(\phi, \phi) - h_{\Lambda}^{A,V}(\phi, \phi)| \leq \Xi_1 h_{\Lambda}^{A,V}(\phi, \phi) + \Xi_2 \|\phi\|^2, \quad \phi \in \mathcal{Q}(h_{\Lambda}^{A,V}), \quad (21)$$

where Ξ_1 and Ξ_2 are given in (19) with a_0 replaced by $|a|$.

To apply Theorem A.1, we choose $s \in (0, \frac{1-\Theta_1}{2})$ so that $\Xi_1 = \frac{2s}{1-\Theta_1} < 1$. Since $h_{\Lambda}^{A,V}$ is symmetric, closed and bounded from below, Theorem A.1 says that $h_{\Lambda}^{A,V}(a)$ is a closed sectorial form defined on $\mathcal{Q}(h_{\Lambda}^{A,V})$. Theorem A.2 then guarantees that there exists a unique m -sectorial operator, denoted by $H_{\Lambda}^a(A, V)$, associated to $h_{\Lambda}^{A,V}(a)$. \square

The next lemma gives an operator equality connecting $H_{\Lambda}^a(A, V)$ and $H_{\Lambda}(A, V)$.

Lemma 4.2. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Suppose $s \in (0, \frac{1-\Theta_1}{2})$ so that $\Xi_1 < 1$. Let the m -sectorial operator $H_{\Lambda}^a(A, V)$ be as in Lemma 4.1. Let*

$$\tilde{H}_{\Lambda}(A, V) = H_{\Lambda}(A, V) + \Xi_1^{-1}\Xi_2, \quad (22)$$

where Ξ_1 and Ξ_2 are given in (19) with a_0 replaced by $|a|$. Then $\tilde{H}_{\Lambda}(A, V)$ is nonnegative and there exists a bounded linear operator B from $L^2(\Lambda)$ to itself with $\|B\| \leq 2\Xi_1$ such that

$$H_{\Lambda}^a(A, V) = H_{\Lambda}(A, V) + \sqrt{\tilde{H}_{\Lambda}(A, V)} B \sqrt{\tilde{H}_{\Lambda}(A, V)}. \quad (23)$$

Proof. Set

$$\begin{aligned} \bar{h}_{\Lambda}^{A,V}(a) &= h_{\Lambda}^{A,V}(a) - h_{\Lambda}^{A,V} \quad \text{on} \quad \mathcal{Q}(h_{\Lambda}^{A,V}), \\ \tilde{h}_{\Lambda}^{A,V} &= h_{\Lambda}^{A,V} + \Xi_1^{-1}\Xi_2 \quad \text{on} \quad \mathcal{Q}(h_{\Lambda}^{A,V}). \end{aligned} \quad (24)$$

Then (21) can be rewritten as

$$|\bar{h}_{\Lambda}^{A,V}(a)(\phi, \phi)| \leq \Xi_1 \tilde{h}_{\Lambda}^{A,V}(\phi, \phi), \quad \phi \in \mathcal{Q}(h_{\Lambda}^{A,V}),$$

which implies that $\tilde{h}_\Lambda^{A,V}$ is a densely defined, symmetric, nonnegative closed sesquilinear form with the associated nonnegative self-adjoint operator $\tilde{H}_\Lambda(A, V)$ defined in (22).

Theorem A.3 then insures that there exists a bounded linear operator B from $L^2(\Lambda)$ to itself with $\|B\| \leq 2\Xi_1$ so that

$$\bar{h}_\Lambda^{A,V}(a)(\psi, \phi) = \left\langle \sqrt{\tilde{H}_\Lambda(A, V)}\psi, B\sqrt{\tilde{H}_\Lambda(A, V)}\phi \right\rangle, \quad \psi, \phi \in \mathcal{Q}(h_\Lambda^{A,V}) = \mathcal{D}\left(\sqrt{\tilde{H}_\Lambda(A, V)}\right). \quad (25)$$

Let

$$\tilde{h}_\Lambda^{A,V}(a) = h_\Lambda^{A,V}(a) + \Xi_1^{-1}\Xi_2 \quad \text{on} \quad \mathcal{Q}(h_\Lambda^{A,V}). \quad (26)$$

Since $h_\Lambda^{A,V}(a)$ is a densely defined closed sectorial form, so does $\tilde{h}_\Lambda^{A,V}(a)$ and the associated m -sectorial operator is given by

$$\tilde{H}_\Lambda^a(A, V) = H_\Lambda^a(A, V) + \Xi_1^{-1}\Xi_2. \quad (27)$$

Considering (24) and (25), we also have

$$\begin{aligned} \tilde{h}_\Lambda^{A,V}(a)(\psi, \phi) &= \tilde{h}_\Lambda^{A,V}(\psi, \phi) + \bar{h}_\Lambda^{A,V}(a)(\psi, \phi) \\ &= \left\langle \sqrt{\tilde{H}_\Lambda(A, V)}\psi, \sqrt{\tilde{H}_\Lambda(A, V)}\phi \right\rangle + \left\langle \sqrt{\tilde{H}_\Lambda(A, V)}\psi, B\sqrt{\tilde{H}_\Lambda(A, V)}\phi \right\rangle \\ &= \left\langle \sqrt{\tilde{H}_\Lambda(A, V)}\psi, (1+B)\sqrt{\tilde{H}_\Lambda(A, V)}\phi \right\rangle, \quad \psi, \phi \in \mathcal{Q}(h_\Lambda^{A,V}). \end{aligned} \quad (28)$$

We claim that

$$\tilde{H}_\Lambda^a(A, V) = \sqrt{\tilde{H}_\Lambda(A, V)}(1+B)\sqrt{\tilde{H}_\Lambda(A, V)}. \quad (29)$$

Let $\phi \in \mathcal{D}(\tilde{H}_\Lambda^a(A, V)) \subset \mathcal{Q}(\tilde{h}_\Lambda^{A,V}(a)) = \mathcal{Q}(h_\Lambda^{A,V})$. We have

$$\tilde{h}_\Lambda^{A,V}(a)(\psi, \phi) = \langle \psi, \tilde{H}_\Lambda^a(A, V)\phi \rangle \quad \text{for all } \psi \in \mathcal{Q}(\tilde{h}_\Lambda^{A,V}(a)) = \mathcal{Q}(h_\Lambda^{A,V}).$$

Comparing this with (28) and recalling the definition of the adjoint of an operator, we see that $\sqrt{\tilde{H}_\Lambda(A, V)}(1+B)\sqrt{\tilde{H}_\Lambda(A, V)}\phi$ exists and is equal to $\tilde{H}_\Lambda^a(A, V)\phi$, which implies that

$$\tilde{H}_\Lambda^a(A, V) \subset \sqrt{\tilde{H}_\Lambda(A, V)}(1+B)\sqrt{\tilde{H}_\Lambda(A, V)},$$

i.e., $\sqrt{\tilde{H}_\Lambda(A, V)}(1+B)\sqrt{\tilde{H}_\Lambda(A, V)}$ extends $\tilde{H}_\Lambda^a(A, V)$. To show (29), it now suffices to show that $\sqrt{\tilde{H}_\Lambda(A, V)}(1+B)\sqrt{\tilde{H}_\Lambda(A, V)}$ is accretive since $\tilde{H}_\Lambda^a(A, V)$ is m -sectorial, so has no proper accretive extension. For any $\psi \in \mathcal{D}\left(\sqrt{\tilde{H}_\Lambda(A, V)}(1+B)\sqrt{\tilde{H}_\Lambda(A, V)}\right) \subset \mathcal{Q}(h_\Lambda^{A,V})$, (28) and (26) give

$$\begin{aligned} &\left\langle \psi, \sqrt{\tilde{H}_\Lambda(A, V)}(1+B)\sqrt{\tilde{H}_\Lambda(A, V)}\psi \right\rangle \\ &= \tilde{h}_\Lambda^{A,V}(a)(\psi, \psi) \\ &= h_\Lambda^{A,V}(a)(\psi, \psi) + \Xi_1^{-1}\Xi_2\|\psi\|^2 \\ &= h_\Lambda^{A,V}(a)(\psi, \psi) - h_\Lambda^{A,V}(\psi, \psi) + h_\Lambda^{A,V}(\psi, \psi) + \Xi_1^{-1}\Xi_2\|\psi\|^2. \end{aligned}$$

It then follows from

$$|\Re(h_{\Lambda}^{A,V}(a)(\psi, \psi) - h_{\Lambda}^{A,V}(\psi, \psi))| \leq |h_{\Lambda}^{A,V}(a)(\psi, \psi) - h_{\Lambda}^{A,V}(\psi, \psi)|$$

and (21) that

$$\begin{aligned} & \Re \left\langle \psi, \sqrt{\tilde{H}_{\Lambda}(A, V)}(1 + B)\sqrt{\tilde{H}_{\Lambda}(A, V)}\psi \right\rangle \\ &= \Re(h_{\Lambda}^{A,V}(a)(\psi, \psi) - h_{\Lambda}^{A,V}(\psi, \psi)) + h_{\Lambda}^{A,V}(\psi, \psi) + \Xi_1^{-1}\Xi_2\|\psi\|^2 \\ &\geq -(\Xi_1 h_{\Lambda}^{A,V}(\phi, \phi) + \Xi_2\|\phi\|^2) + h_{\Lambda}^{A,V}(\psi, \psi) + \Xi_1^{-1}\Xi_2\|\psi\|^2 \\ &= (1 - \Xi_1)(h_{\Lambda}^{A,V}(\psi, \psi) + \Xi_1^{-1}\Xi_2\|\psi\|^2) \\ &\geq 0, \end{aligned}$$

since Ξ_1 is taken to be less than 1 and $h_{\Lambda}^{A,V} + \Xi_1^{-1}\Xi_2$ is nonnegative by (21). This shows that $\sqrt{\tilde{H}_{\Lambda}(A, V)}(1 + B)\sqrt{\tilde{H}_{\Lambda}(A, V)}$ is accretive and, thus, (29) holds. Obviously, (23) is equivalent to (29) due to (22) and (27). This completes the proof. \square

The last lemma bridges the resolvent set of $H_{\Lambda}(A, V)$ and that of $H_{\Lambda}^a(A, V)$. Before stating the result, we make following assumptions.

Pick and fix $\lambda_0 < \min\{0, E_0\}$, where E_0 is defined in (6). This number is picked to be of technical use. The advantage is that $H_{\Lambda}(A, V) - \lambda_0$ is strictly positive so that $(H_{\Lambda}(A, V) - \lambda_0)^{\frac{1}{2}}$ is well-defined and boundedly invertible, as opposed to the ill-posedness of the fractional power of $H_{\Lambda}(A, V) - z$, which may cause some troubles.

Let

$$c_{z, \lambda_0} = \left\| \frac{\lambda - \lambda_0}{\lambda - z} \right\|_{L^{\infty}(\sigma(H_{\Lambda}(A, V)))}. \quad (30)$$

Suppose that $s > 0$ satisfies

$$s < \frac{1 - \Theta_1}{4c_{z, \lambda_0}} \quad (31)$$

and $a_0 > 0$ satisfies

$$a_0^2 \leq \frac{2s\lambda_0}{\Theta_1 - 1} \frac{1}{c_s} \quad (32)$$

or

$$\frac{2s\lambda_0}{\Theta_1 - 1} \frac{1}{c_s} \leq a_0^2 < \left(\frac{-\lambda_0}{10c_{z, \lambda_0}} - \frac{8s\lambda_0}{5(1 - \Theta_1)} \right) \frac{1}{c_s}, \quad (33)$$

where Θ_1, Θ_2 are given in (4) and c_s is given in (18).

Remark 4.3. Note that assumptions (32) and (33) can be considered together to form a more general one, but we consider them separately anyway for the following two reasons.

- (i) The first reason is about the conditions giving rise to (32) and the first inequality in (33). In the proof of Lemma 4.4 below, we need conditions on s and a_0 to insure

$$2\Xi_1 c_{z, \lambda_0} \left\| \frac{\lambda + \Xi_1^{-1}\Xi_2}{\lambda - \lambda_0} \right\|_{L^{\infty}(\sigma(H_{\Lambda}(A, V)))} < 1,$$

i.e., (37), where the quantity $\left\| \frac{\lambda + \Xi_1^{-1} \Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))}$ appears. It's easy to see that

$$\left\| \frac{\lambda + \Xi_1^{-1} \Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} = \begin{cases} 1, & \text{if } \Xi_1^{-1} \Xi_2 \leq -\lambda_0, \\ \frac{\inf \sigma(H_\Lambda(A, V)) + \Xi_1^{-1} \Xi_2}{\inf \sigma(H_\Lambda(A, V)) - \lambda_0}, & \text{if } \Xi_1^{-1} \Xi_2 \geq -\lambda_0. \end{cases}$$

Moreover, (32) and the first inequality in (33) correspond to $\Xi_1^{-1} \Xi_2 \leq -\lambda_0$ and $\Xi_1^{-1} \Xi_2 \geq -\lambda_0$, respectively.

(ii) The second reason is that (33) provides a lower bound for a_0 , and in turn, an upper bound for $e^{-a_0|\beta-\gamma|}$, which is important in Section 5 because we need such an upper bound, of course after being simplified, to estimate some integrals there.

Lemma 4.4. Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Let $z \in \rho(H_\Lambda(A, V))$, the resolvent set of $H_\Lambda(A, V)$. Suppose that $s > 0$ and $a \in \mathbb{R}^d$ satisfies (31) and $|a| = a_0$, respectively, where $a_0 > 0$ obeys (32) or (33). Then $H_\Lambda^a(A, V) - z$ is invertible, i.e., $z \in \rho(H_\Lambda^a(A, V))$, the resolvent set of $H_\Lambda^a(A, V)$. In other words, $\rho(H_\Lambda(A, V)) \subset \rho(H_\Lambda^a(A, V))$.

Proof. By (23), we have

$$\begin{aligned} H_\Lambda^a(A, V) - z &= H_\Lambda(A, V) - z + \sqrt{\tilde{H}_\Lambda(A, V)} B \sqrt{\tilde{H}_\Lambda(A, V)} \\ &= (H_\Lambda(A, V) - \lambda_0)^{\frac{1}{2}} (U + V) (H_\Lambda(A, V) - \lambda_0)^{\frac{1}{2}}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} U &= (H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} (H_\Lambda(A, V) - z) (H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} \\ &= (H_\Lambda(A, V) - z) (H_\Lambda(A, V) - \lambda_0)^{-1} \end{aligned}$$

and

$$V = (H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} \sqrt{\tilde{H}_\Lambda(A, V)} B \sqrt{\tilde{H}_\Lambda(A, V)} (H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}}.$$

Since $(H_\Lambda(A, V) - \lambda_0)^{\frac{1}{2}}$ is invertible, $H_\Lambda^a(A, V) - z$ is invertible if and only if $U + V$ is invertible.

We claim that $U + V$ is invertible under the assumption of the lemma with

$$\|(U + V)^{-1}\| \leq \begin{cases} \frac{c_{z, \lambda_0}(1 - \Theta_1)}{1 - \Theta_1 - 4sc_{z, \lambda_0}}, & \text{if } a_0 \text{ satisfies (32),} \\ \frac{c_{z, \lambda_0} \lambda_0}{\lambda_0 + (8\lambda_0 \Xi_1 + 10\Xi_2)c_{z, \lambda_0}}, & \text{if } a_0 \text{ satisfies (33).} \end{cases} \quad (35)$$

Obviously, U is bounded and invertible with

$$U^{-1} = (H_\Lambda(A, V) - \lambda_0)(H_\Lambda(A, V) - z)^{-1}.$$

Recall that $\tilde{H}_\Lambda(A, V) = H_\Lambda(A, V) + \Xi_1^{-1} \Xi_2 \geq 0$. Since the function $\sqrt{\frac{\lambda + \Xi_1^{-1} \Xi_2}{\lambda - \lambda_0}}$ is bounded on $\sigma(H_\Lambda(A, V))$, both $(H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} \sqrt{\tilde{H}_\Lambda(A, V)}$ and $\sqrt{\tilde{H}_\Lambda(A, V)} (H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}}$ are

bounded, which implies that V is bounded. Then, by stability of bounded invertibility (see [23, Theorem IV.1.16]), it suffices to require that $\|V\|\|U^{-1}\| < 1$. In which case, $U + V$ is invertible with

$$\|(U + V)^{-1}\| \leq \frac{\|U^{-1}\|}{1 - \|V\|\|U^{-1}\|}. \quad (36)$$

Since $\|U^{-1}\| \leq c_{z, \lambda_0}$ and

$$\|V\| \leq \|B\| \left\| \sqrt{\frac{\lambda + \Xi_1^{-1} \Xi_2}{\lambda - \lambda_0}} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))}^2 \leq 2\Xi_1 \left\| \frac{\lambda + \Xi_1^{-1} \Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))},$$

it is sufficient to require that

$$2\Xi_1 c_{z, \lambda_0} \left\| \frac{\lambda + \Xi_1^{-1} \Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} < 1. \quad (37)$$

It is justified in Appendix B that if s and a are as in the assumptions of the current lemma, then (37) is satisfied, which then implies that $U + V$, and hence $H_\Lambda^a(A, V) - z$, is invertible. Finally, (35) follows from (36) and Appendix B. \square

We proceed to prove Theorem 1.1. Since the proof in the case $n \geq 2$ is based on the proof in the case $n = 1$, we divide Theorem 1.1 into two parts according to $n = 1$ and $n \geq 2$. Moreover, we restate the theorem in the cases $n = 1$ and $n \geq 2$ as Theorem 4.5 and Theorem 4.6 below, respectively.

Theorem 4.5. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Suppose $p > \frac{d}{2}$. Suppose that $s > 0$ and $a_0 > 0$ satisfy (31) and (32) or (33). Let $z \in \rho(H_\Lambda(A, V))$, the resolvent set of $H_\Lambda(A, V)$. Then, for any $\beta, \gamma \in \mathbb{R}^d$,*

$$\begin{aligned} & \|\chi_\beta(H_\Lambda(A, V) - z)^{-1} \chi_\gamma\|_{\mathcal{J}_p} \\ & \leq \begin{cases} \frac{C_{p, \lambda_0} c_{z, \lambda_0} (1 - \Theta_1)}{1 - \Theta_1 - 4s c_{z, \lambda_0}} e^{\sqrt{d} a_0} e^{-a_0 |\beta - \gamma|}, & \text{if } a_0 \text{ satisfies (32),} \\ \frac{C_{p, \lambda_0} c_{z, \lambda_0} \lambda_0}{\lambda_0 + (8\lambda_0 \Xi_1 + 10\Xi_2) c_{z, \lambda_0}} e^{\sqrt{d} a_0} e^{-a_0 |\beta - \gamma|}, & \text{if } a_0 \text{ satisfies (33),} \end{cases} \end{aligned} \quad (38)$$

where $C_{p, \lambda_0} > 0$ depends only on p and λ_0 .

Proof. By Lemma 4.4 and the operator equality (34), we have

$$\chi_\beta(H_\Lambda^a(A, V) - z)^{-1} \chi_\gamma = \chi_\beta(H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} (U + V)^{-1} (H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} \chi_\gamma.$$

Since the function $(\lambda - \lambda_0)^{-\frac{1}{2}}$ satisfies (7) with $\alpha = \frac{1}{2}$, $\frac{1}{2} > \frac{d}{2 \cdot 2p}$ and $2p > d \geq 2$, Theorem 3.1 insures that both $\chi_\beta(H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}}$ and $(H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} \chi_\gamma$ are in \mathcal{J}_{2p} with \mathcal{J}_{2p} -norm only depending on p and λ_0 . It then follows that $\chi_\beta(H_\Lambda^a(A, V) - z)^{-1} \chi_\gamma \in \mathcal{J}_p$ with

$$\begin{aligned} & \|\chi_\beta(H_\Lambda^a(A, V) - z)^{-1} \chi_\gamma\|_{\mathcal{J}_p} \\ & \leq \|\chi_\beta(H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}}\|_{\mathcal{J}_{2p}} \|(U + V)^{-1}\| \cdot \|(H_\Lambda(A, V) - \lambda_0)^{-\frac{1}{2}} \chi_\gamma\|_{\mathcal{J}_{2p}} \\ & \leq \begin{cases} \frac{C_{p, \lambda_0} c_{z, \lambda_0} (1 - \Theta_1)}{1 - \Theta_1 - 4s c_{z, \lambda_0}}, & \text{if } a_0 \text{ satisfies (32),} \\ \frac{C_{p, \lambda_0} c_{z, \lambda_0} \lambda_0}{\lambda_0 + (8\lambda_0 \Xi_1 + 10\Xi_2) c_{z, \lambda_0}}, & \text{if } a_0 \text{ satisfies (33).} \end{cases} \end{aligned} \quad (39)$$

where (35) is used and $C_{p,\lambda_0} > 0$ only depends on p and λ_0 . Considering (14), we obtain

$$\begin{aligned}
& \|\chi_\beta(H_\Lambda(A, V) - z)^{-1}\chi_\gamma\|_{\mathcal{J}_p} \\
&= \|\chi_\beta e^{-a \cdot x}(H_\Lambda^a(A, V) - z)^{-1}e^{a \cdot x}\chi_\gamma\|_{\mathcal{J}_p} \\
&= \|e^{-a \cdot (\beta - \gamma)}(e^{-a \cdot (x - \beta)}\chi_\beta)(\chi_\beta(H_\Lambda^a(A, V) - z)^{-1}\chi_\gamma)(\chi_\gamma e^{a \cdot (x - \gamma)})\|_{\mathcal{J}_p} \\
&\leq \|\chi_\beta(H_\Lambda^a(A, V) - z)^{-1}\chi_\gamma\|_{\mathcal{J}_p} \|e^{-a \cdot (x - \beta)}\chi_\beta\| \cdot \|\chi_\gamma e^{a \cdot (x - \gamma)}\| e^{-a \cdot (\beta - \gamma)} \\
&\leq \begin{cases} \frac{C_{p,\lambda_0} c_{z,\lambda_0} (1 - \Theta_1)}{1 - \Theta_1 - 4s c_{z,\lambda_0}} e^{\sqrt{d}|a|} e^{-a \cdot (\beta - \gamma)}, & \text{if } a_0 \text{ satisfies (32),} \\ \frac{C_{p,\lambda_0} c_{z,\lambda_0} \lambda_0}{\lambda_0 + (8\lambda_0 \Xi_1 + 10\Xi_2) c_{z,\lambda_0}} e^{\sqrt{d}|a|} e^{-a \cdot (\beta - \gamma)}, & \text{if } a_0 \text{ satisfies (33),} \end{cases}
\end{aligned}$$

where we have used (39) and the fact that both $\|e^{-a \cdot (x - \beta)}\chi_\beta\|$ and $\|\chi_\gamma e^{a \cdot (x - \gamma)}\|$ are bounded by $e^{\frac{\sqrt{d}}{2}|a|}$. By choosing $a = a_0|\beta - \gamma|^{-1}(\beta - \gamma)$, we find (38). This completes the proof. \square

Theorem 4.6. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_\pm(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Suppose $p > \frac{d}{2n}$ with $n \in \mathbb{N}$ and $n \geq 2$. Let $z \in \rho(H_\Lambda(A, V))$. Suppose that $s > 0$ and $a_0 > 0$ satisfy (31) and (32) or (33). Then, for any $\delta \in (0, 1)$ and any $\beta, \gamma \in \mathbb{R}^d$, there holds*

$$\begin{aligned}
& \|\chi_\beta(H_\Lambda(A, V) - z)^{-n}\chi_\gamma\|_{\mathcal{J}_p} \\
&\leq \begin{cases} \left(\frac{C_{p,n,\lambda_0} c_{\delta,a_0} c_{z,\lambda_0} (1 - \Theta_1)}{1 - \Theta_1 - 4s c_{z,\lambda_0}} \right)^{n-1} e^{(n-1)\sqrt{d}a_0} e^{-\delta a_0|\beta - \gamma|}, & \text{if } a_0 \text{ satisfies (32),} \\ \left(\frac{C_{p,n,\lambda_0} c_{\delta,a_0} c_{z,\lambda_0} \lambda_0}{\lambda_0 + (8\lambda_0 \Xi_1 + 10\Xi_2) c_{z,\lambda_0}} \right)^{n-1} e^{(n-1)\sqrt{d}a_0} e^{-\delta a_0|\beta - \gamma|}, & \text{if } a_0 \text{ satisfies (33),} \end{cases} \quad (40)
\end{aligned}$$

where $C_{p,n,\lambda_0} > 0$ only depends on p, n and λ_0 , c_{z,λ_0} is given in (30) and $c_{\delta,a_0} = \sum_{\alpha \in \mathbb{Z}^d} e^{-(1-\delta)a_0|\alpha|} < \infty$.

Proof. Write

$$\chi_\beta(H_\Lambda(A, V) - z)^{-n}\chi_\gamma = \sum_{\substack{\alpha_j \in \mathbb{Z}^d \\ j=1, \dots, n-1}} R_{\beta, \alpha_1} R_{\alpha_1, \alpha_2} \cdots R_{\alpha_{n-2}, \alpha_{n-1}} R_{\alpha_{n-1}, \gamma},$$

where $R_{\beta, \alpha_1} = \chi_\beta(H_\Lambda(A, V) - z)^{-1}\chi_{\alpha_1}$, $R_{\alpha_j, \alpha_{j+1}} = \chi_{\alpha_j}(H_\Lambda(A, V) - z)^{-1}\chi_{\alpha_{j+1}}$, $j = 1, \dots, n-2$ and $R_{\alpha_{n-1}, \gamma} = \chi_{\alpha_{n-1}}(H_\Lambda(A, V) - z)^{-1}\chi_\gamma$. Since $pn > \frac{d}{2}$ by assumption, Theorem 4.5 says that

$$\begin{aligned}
& \|\chi_x(H_\Lambda(A, V) - z)^{-1}\chi_y\|_{\mathcal{J}_{pn}} \\
&\leq \begin{cases} \frac{C_{p,n,\lambda_0} c_{z,\lambda_0} (1 - \Theta_1)}{1 - \Theta_1 - 4s c_{z,\lambda_0}} e^{\sqrt{d}a_0} e^{-a_0|\beta - \gamma|}, & \text{if } a_0 \text{ satisfies (32),} \\ \frac{C_{p,n,\lambda_0} c_{z,\lambda_0} \lambda_0}{\lambda_0 + (8\lambda_0 \Xi_1 + 10\Xi_2) c_{z,\lambda_0}} e^{\sqrt{d}a_0} e^{-a_0|\beta - \gamma|}, & \text{if } a_0 \text{ satisfies (33),} \end{cases} \quad \forall x, y \in \mathbb{R}^d,
\end{aligned}$$

where $C_{p,n,\lambda_0} > 0$ only depends on p , n and λ_0 . By Hölder's inequality for trace ideals (see [36, Theorem 2.8]), the result of the corollary follows from

$$\begin{aligned} & \sum_{\substack{\alpha_j \in \mathbb{Z}^d \\ j=1,\dots,n-1}} e^{-a_0|\beta-\alpha_1|} e^{-a_0|\alpha_1-\alpha_2|} \dots e^{-a_0|\alpha_{n-2}-\alpha_{n-1}|} e^{-a_0|\alpha_{n-1}-\beta|} \\ & \leq c_{\delta,a_0}^{n-1} e^{-\delta a_0|\beta-\gamma|}, \quad \forall \delta \in (0,1), \end{aligned} \tag{41}$$

where $c_{\delta,a_0} = \sum_{\alpha \in \mathbb{Z}^d} e^{-(1-\delta)a_0|\alpha|} < \infty$.

To complete the proof, we show (41). Pick and fix any $\delta \in (0,1)$. First, we have from the triangular inequality and Cauchy's inequality

$$\begin{aligned} & \sum_{\alpha_1 \in \mathbb{Z}^d} e^{-a_0|\beta-\alpha_1|} e^{-a_0|\alpha_1-\alpha_2|} \\ & = \sum_{\alpha_1 \in \mathbb{Z}^d} e^{-(1-\delta)a_0|\beta-\alpha_1|} e^{-\delta a_0(|\beta-\alpha_1|+|\alpha_1-\alpha_2|)} e^{-(1-\delta)a_0|\alpha_1-\alpha_2|} \\ & \leq e^{-\delta a_0|\beta-\alpha_2|} \sum_{\alpha_1 \in \mathbb{Z}^d} e^{-(1-\delta)a_0|\beta-\alpha_1|} e^{-(1-\delta)a_0|\alpha_1-\alpha_2|} \\ & \leq e^{-\delta a_0|\beta-\alpha_2|} \left(\sum_{\alpha_1 \in \mathbb{Z}^d} e^{-2(1-\delta)a_0|\beta-\alpha_1|} \right)^{\frac{1}{2}} \left(\sum_{\alpha_1 \in \mathbb{Z}^d} e^{-2(1-\delta)a_0|\alpha_1-\alpha_2|} \right)^{\frac{1}{2}} \\ & \leq c_{\delta,a_0} e^{-\delta a_0|\beta-\alpha_2|}, \end{aligned}$$

where $c_{\delta,a_0} = \sum_{\alpha_1 \in \mathbb{Z}^d} e^{-(1-\delta)a_0|\alpha_1|} \geq \sum_{\alpha_1 \in \mathbb{Z}^d} e^{-2(1-\delta)a_0|\alpha_1|}$. Next, by the above estimate and the triangular inequality,

$$\begin{aligned} & \sum_{\alpha_2 \in \mathbb{Z}^d} \sum_{\alpha_1 \in \mathbb{Z}^d} e^{-a_0|\beta-\alpha_1|} e^{-a_0|\alpha_1-\alpha_2|} e^{-a_0|\alpha_2-\alpha_3|} \\ & \leq c_{\delta,a_0} \sum_{\alpha_2 \in \mathbb{Z}^d} e^{-\delta a_0|\beta-\alpha_2|} e^{-a_0|\alpha_2-\alpha_3|} \\ & = c_{\delta,a_0} \sum_{\alpha_2 \in \mathbb{Z}^d} e^{-\delta a_0|\beta-\alpha_2|} e^{-\delta a_0|\alpha_2-\alpha_3|} e^{-(1-\delta)a_0|\alpha_2-\alpha_3|} \\ & \leq c_{\delta,a_0} e^{-\delta a_0|\beta-\alpha_3|} \sum_{\alpha_2 \in \mathbb{Z}^d} e^{-(1-\delta)a_0|\alpha_2-\alpha_3|} \\ & = c_{\delta,a_0}^2 e^{-\delta a_0|\beta-\alpha_3|}. \end{aligned}$$

By induction, we find (41). This completes the proof. \square

5 The Operator Kernel Estimate in Trace Ideals

In this section, we study the operator kernel estimate in trace-class norms. More precisely, we prove polynomial decay, in trace ideals, of operators

$$\chi_\beta f(H_\Lambda(A, V)) \chi_\gamma, \quad \beta, \gamma \in \mathbb{R}^d$$

in terms of $|\beta - \gamma|$ for $f \in \mathcal{S}(\mathbb{R})$, the Schwartz space. The main result in this section is stated in Theorem 1.2, whose proof is based on Theorem 1.1 (in fact, Theorem 4.5) and the Helffer-Sjöstrand formula (see [22]), which is defined for a much larger class of slowly decreasing smooth functions on \mathbb{R} , denoted by \mathcal{A} . See Appendix C for the definition of \mathcal{A} and the Helffer-Sjöstrand formula.

Before proving Theorem 1.2, we first simplify the second estimate in (38) by adding more conditions so that this estimate can be easily used. Our idea is as follows: by the Helffer-Sjöstrand formula (65), we have for any $f \in \mathcal{S}(\mathbb{R})$,

$$\chi_\beta f(H_\Lambda(A, V)) \chi_\gamma = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} \chi_\beta(H_\Lambda(A, V) - z)^{-1} \chi_\gamma dudv, \quad \forall n \geq 1.$$

Therefore, by (64),

$$\begin{aligned} & \|\chi_\beta f(H_\Lambda(A, V)) \chi_\gamma\|_{\mathcal{J}_p} \\ & \leq \frac{C}{\pi} \sum_{r=0}^n \frac{1}{r!} \int_U |f^{(r)}(u)| \frac{|v|^r}{\langle u \rangle} \|\chi_\beta(H_\Lambda(A, V) - z)^{-1} \chi_\gamma\|_{\mathcal{J}_p} dudv \\ & \quad + \frac{1}{2\pi n!} \int_V |f^{(n+1)}(u)| |v|^n \|\chi_\beta(H_\Lambda(A, V) - z)^{-1} \chi_\gamma\|_{\mathcal{J}_p} dudv \end{aligned} \quad (42)$$

for any $n \geq 1$. Clearly, in order to estimate the integrals on the right hand side of (42), we need (38), and we pick the second estimate in (38) because it provides a lower bound for a_0 , which in turn provides an upper bound for the exponential term. However, this estimate is too rough to deal with since many parameters in the bound depend on z . To simplify it, we put more conditions on s and a_0 .

We suppose that $s > 0$ and $a_0 > 0$ satisfy

$$s < \frac{1 - \Theta_1}{44c_{z, \lambda_0}} \quad (43)$$

and

$$\frac{2\lambda_0 s}{\Theta_1 - 1} \frac{1}{c_s} \leq a_0^2 \leq \frac{4\lambda_0 s}{\Theta_1 - 1} \frac{1}{c_s}, \quad (44)$$

where $c_s = \frac{2s\Theta_2}{1-\Theta_1} + \frac{1}{2}(\frac{1}{s} + s)$ is the same as in (18). We remark that the constant 44 in (43) is nothing special, and it can be replaced by any other one as long as the second condition in (45) below is satisfied. The intuitive interpretation of these conditions is that we make $2\Xi_1 c_{z, \lambda_0}$ smaller and bound $\Xi_1^{-1} \Xi_2$ by $-2\lambda_0$ from above. Indeed, if (43) and (44) are satisfied, then

$$2\Xi_1 c_{z, \lambda_0} < \frac{1}{11} \quad \text{and} \quad -\lambda_0 \Xi_1 \leq \Xi_2 \leq -2\lambda_0 \Xi_1 < \frac{-\lambda_0}{10c_{z, \lambda_0}} - \frac{4}{5} \lambda_0 \Xi_1. \quad (45)$$

As opposed to (62), the current conditions are stronger. Also, we can verify directly that (43) and (44) are stronger than (31) and (33). Hence, under the assumptions of (43) and (44), the second estimate in (38) holds and we can simplify it to get

$$\|\chi_\beta(H_\Lambda(A, V) - z)^{-1} \chi_\gamma\|_{\mathcal{J}_p} \leq C_{p, \lambda_0} c_{z, \lambda_0} e^{\sqrt{d}a_0} e^{-a_0|\beta - \gamma|}, \quad (46)$$

where $C_{p,\lambda_0} > 0$ depends only on p and λ_0 . Furthermore, we simplify (44) to get

$$-\lambda_0 \left[\Theta_2 + \frac{1-\Theta_1}{4} \left(\frac{1}{s^2} + 1 \right) \right]^{-1} \leq a_0^2 \leq -2\lambda_0 \left[\Theta_2 + \frac{1-\Theta_1}{4} \left(\frac{1}{s^2} + 1 \right) \right]^{-1}. \quad (47)$$

Since $c_{z,\lambda_0} \geq 1$, (43) says that s is bounded by a constant independent of z . It then follows from (47) that a_0 is bounded by a constant independent of z , which implies that $e^{\sqrt{d}a_0}$ is bounded by a constant independent of z . Hence, (46) is further simplified to

$$\|\chi_\beta(H_\Lambda(A, V) - z)^{-1}\chi_\gamma\|_{\mathcal{J}_p} \leq C_{p,\lambda_0} c_{z,\lambda_0} e^{-a_0|\beta-\gamma|}, \quad (48)$$

where $C_{p,\lambda_0} > 0$ only depends on p and λ_0 .

Note that the lower bound of a_0 is not very easy to handle because of the uncertainty of s and the quantity c_{z,λ_0} . To find a simpler lower bound for a_0 , we first fix some s , say $s = \frac{1-\Theta_1}{66c_{z,\lambda_0}}$, and then we give explicit bound for c_{z,λ_0} with $z = u + iv$ under assumptions $(u, v) \in U$ and $(u, v) \in V$, respectively. Recall that $c_{z,\lambda_0} = \left\| \frac{\lambda - \lambda_0}{\lambda - z} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))}$, $U = \{(u, v) \in \mathbb{R}^2 | \langle u \rangle < |v| < 2\langle u \rangle\}$ and $V = \{(u, v) \in \mathbb{R}^2 | 0 < |v| < 2\langle u \rangle\}$.

Lemma 5.1. *Let $z \in \rho(H_\Lambda(A, V))$, $s = \frac{1-\Theta_1}{66c_{z,\lambda_0}}$ and $a_0 > 0$ satisfy (44). Then, with $z = u + iv$,*

$$\|\chi_\beta(H_\Lambda(A, V) - z)^{-1}\chi_\gamma\|_{\mathcal{J}_p} \leq \begin{cases} C_{p,\lambda_0} |v| e^{-\frac{C_{\lambda_0}}{|v|}|\beta-\gamma|}, & \text{if } (u, v) \in U, \\ C_{p,\lambda_0} \frac{\langle u \rangle}{|v|} e^{-C_{\lambda_0} \frac{|v|}{\langle u \rangle}|\beta-\gamma|}, & \text{if } (u, v) \in V, \end{cases} \quad (49)$$

where $C_{\lambda_0} > 0$ depends only on λ_0 and $C_{p,\lambda_0} > 0$ depends only on p and λ_0 .

Proof. By the first inequality in (47) and the assumption $s = \frac{1-\Theta_1}{66c_{z,\lambda_0}}$, we have

$$a_0^2 \geq -\lambda_0 \left[\Theta_2 + \frac{1-\Theta_1}{4} \left(\frac{1}{s^2} + 1 \right) \right]^{-1} = \frac{-\lambda_0 \tilde{C}}{c_{z,\lambda_0}^2 + \tilde{C}} \quad (50)$$

for some $\tilde{C} > 0$ and $\tilde{C} > 0$.

Let $(u, v) \in U$. For any $\lambda \in \sigma(H_\Lambda(A, V))$, $|\lambda - z| \geq \text{dist}(z, \sigma(H_\Lambda(A, V))) \geq |v| > \langle u \rangle \geq 1$, which implies that

$$c_{z,\lambda_0} \leq \left\| 1 + \frac{|z - \lambda_0|}{|\lambda - z|} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} \leq 1 + |z| - \lambda_0 \leq 1 - \lambda_0 + \sqrt{2}|v| \leq C_{\lambda_0}|v|$$

and then

$$a_0 \geq \sqrt{\frac{-\lambda_0 \tilde{C}}{c_{z,\lambda_0}^2 + \tilde{C}}} \geq \frac{C_{\lambda_0}}{|v|}, \quad (51)$$

where the fact $|v| \geq 1$ is used.

Let $(u, v) \in V$. Then

$$c_{z,\lambda_0} \leq \left\| 1 + \frac{|z - \lambda_0|}{|\lambda - z|} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} \leq 1 + \frac{|z| - \lambda_0}{|v|} \leq \frac{5\langle u \rangle - \lambda_0}{|v|} \leq C_{\lambda_0} \frac{\langle u \rangle}{|v|},$$

which, together with (50), implies that

$$a_0 \geq C_{\lambda_0} \frac{|v|}{\langle u \rangle}, \quad (52)$$

where the fact $\frac{\langle u \rangle}{|v|} > \frac{1}{2}$ is used.

By means of (48), (51) and (52), we find (49). \square

We now restate and prove Theorem 1.2.

Theorem 5.2. *Let $A \in \mathcal{H}_{loc}(\mathbb{R}^d)$, $V \in \mathcal{K}_{\pm}(\mathbb{R}^d)$ and $\Lambda \subset \mathbb{R}^d$ open. Suppose $p > \frac{d}{2}$. For $z \in \rho(H_{\Lambda}(A, V))$, we let $s = \frac{1-\Theta_1}{66c_{z,\lambda_0}}$ and suppose that $a_0 > 0$ satisfies (44), i.e.,*

$$\frac{2\lambda_0 s}{\Theta_1 - 1} \frac{1}{c_s} \leq a_0^2 \leq \frac{4\lambda_0 s}{\Theta_1 - 1} \frac{1}{c_s}, \quad (53)$$

where c_s is given in (18). Then, for any $f \in \mathcal{S}(\mathbb{R})$ and any $k \in \mathbb{Z}$ with $k \geq 1$,

$$\|\chi_{\beta} f(H_{\Lambda}(A, V)) \chi_{\gamma}\|_{\mathcal{J}_p} \leq C_{p,\lambda_0,k,f} |\beta - \gamma|^{-k}, \quad \forall \beta, \gamma \in \mathbb{R}^d, \quad (54)$$

where $C_{p,\lambda_0,k,f} > 0$ depends only on p , λ_0 , k and f .

Remark 5.3. *The constant $s = \frac{1-\Theta_1}{66c_{z,\lambda_0}}$ in the above theorem is nothing special, and thus, can be replaced by any other one satisfying required conditions.*

Proof of Theorem 5.2. Fix any $k \in \mathbb{Z}$ with $k \geq 1$ and let $n = k + 1$ in (42). Since the function $\theta(t) = e^{-t} t^k$, $t \geq 0$ attains its global maximum at $t = k$, we have

$$e^{-t} \leq e^{-k} k^k t^{-k}. \quad (55)$$

Applying (55) to $t = \frac{C_{\lambda_0}}{|v|} |\beta - \gamma|$ and $t = C_{\lambda_0} \frac{|v|}{\langle u \rangle} |\beta - \gamma|$, respectively, we obtain

$$e^{-\frac{C_{\lambda_0}}{|v|} |\beta - \gamma|} \leq \frac{e^{-k} k^k}{C_{\lambda_0}^k |\beta - \gamma|^k} |v|^k \quad (56)$$

and

$$e^{-C_{\lambda_0} \frac{|v|}{\langle u \rangle} |\beta - \gamma|} \leq \frac{e^{-k} k^k}{C_{\lambda_0}^k |\beta - \gamma|^k} \frac{\langle u \rangle^k}{|v|^k}, \quad (57)$$

respectively.

We now use (56) and (57) to estimate the integrals in (42). By the first estimate in (49) and (56), we have for some $C_{p,\lambda_0,k,f} > 0$,

$$\begin{aligned} & \sum_{r=0}^{k+1} \frac{1}{r!} \int_U |f^{(r)}(u)| \frac{|v|^r}{\langle u \rangle} \|\chi_{\beta}(H_{\Lambda}(A, V) - z)^{-1} \chi_{\gamma}\|_{\mathcal{J}_p} du dv \\ & \leq \frac{C_{p,\lambda_0} e^{-k} k^k}{C_{\lambda_0}^k |\beta - \gamma|^k} \sum_{r=0}^{k+1} \frac{1}{r!} \int_U |f^{(r)}(u)| \frac{|v|^{r+k+1}}{\langle u \rangle} du dv \\ & = \frac{C_{p,\lambda_0} e^{-k} k^k}{C_{\lambda_0}^k |\beta - \gamma|^k} \sum_{r=0}^{k+1} \frac{1}{r!} \frac{2^{r+k+3} - 2}{r + k + 2} \int_{\mathbb{R}} |f^{(r)}(u)| \langle u \rangle^{r+k+1} du \\ & \leq C_{p,\lambda_0,k,f} |\beta - \gamma|^{-k}, \end{aligned}$$

where the fact $f \in \mathcal{S}(\mathbb{R})$, so the integrals are convergent, is used.

Similarly, by the second estimate in (49) and (57),

$$\begin{aligned}
& \frac{1}{2\pi(k+1)!} \int_V |f^{(n+1)}(u)| |v|^n \|\chi_\beta(H_\Lambda(A, V) - z)^{-1} \chi_\gamma\|_{\mathcal{J}_p} du dv \\
& \leq \frac{C_{p,\lambda_0} e^{-k} k^k}{2\pi(k+1)! C_{\lambda_0}^k |\beta - \gamma|^k} \int_V |f^{(k+2)}(u)| \langle u \rangle^{k+1} du dv \\
& = \frac{4C_{p,\lambda_0} e^{-k} k^k}{2\pi(k+1)! C_{\lambda_0}^k |\beta - \gamma|^k} \int_{\mathbb{R}} |f^{(k+2)}(u)| \langle u \rangle^{k+2} du \\
& \leq C_{p,\lambda_0,k,f} |\beta - \gamma|^{-k}.
\end{aligned}$$

Consequently, for any $f \in \mathcal{S}(\mathbb{R})$, there exists $C_{p,\lambda_0,k,f} > 0$ so that

$$\|\chi_\beta f(H_\Lambda(A, V)) \chi_\gamma\|_{\mathcal{J}_p} \leq C_{p,\lambda_0,k,f} |\beta - \gamma|^{-k}, \quad \forall \beta, \gamma \in \mathbb{R}^d.$$

This proves Theorem 5.2. □

Acknowledgments

It is a pleasure to thank Wenxian Shen for her support during the work of this paper.

A Sectorial Form and m -Sectorial Operator

In this section, we review some results about sectorial form and m -sectorial operator used in the above sections. The material is chosen from [23]. Also see [12].

Let \mathcal{H} be a separable Hilbert space and $h(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ be a sesquilinear form. It is called *sectorial* if there exist $\gamma \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ so that

$$h(u, u) \in \{z \in \mathbb{C} \mid |\arg(z - \gamma)| \leq \theta\} \text{ for any } u \in \mathcal{Q}(h) \text{ with } \|u\| = 1,$$

where $\mathcal{Q}(h)$ is the form domain of h . In particular, any symmetric sesquilinear form bounded from below is sectorial. For relatively bounded perturbation, we have (see [23, Theorem VI.1.33])

Theorem A.1. *Let h be a sectorial form and h' be h -bounded, i.e., $\mathcal{Q}(h) \subset \mathcal{Q}(h')$ and there exist nonnegative constants a and b such that*

$$|h'(u, u)| \leq ah(u, u) + b\|u\|^2 \text{ for any } u \in \mathcal{Q}(h).$$

If $a < 1$, then $h + h'$ is sectorial. $h + h'$ is closable or closed if and only if h is closable or closed, respectively.

Let $H : \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator with domain $\mathcal{D}(H)$. H is said to be *accretive* if $\Re \langle u, Hu \rangle \geq 0$ for all $u \in \mathcal{D}(H)$. It is said to be *m -accretive* if for any $z \in \mathbb{C}$ with $\Re z > 0$, there hold

$$(H + z)^{-1} \in \mathcal{L}(\mathcal{H}) \quad \text{and} \quad \|(H + z)^{-1}\| \leq \frac{1}{\Re z},$$

where $\mathcal{L}(\mathcal{H})$ denotes the space of all bounded linear operators on \mathcal{H} . It's not hard to see that m -accretive operator is maximal accretive in the sense that it is accretive and has no proper accretive extension. If there are $\gamma \in \mathbb{R}$ and $\theta \in [0, \frac{\pi}{2})$ so that

$$\langle u, Hu \rangle \in \{z \in \mathbb{C} \mid |\arg(z - \gamma)| \leq \theta\} \text{ for any } u \in \mathcal{D}(H) \text{ with } \|u\| = 1,$$

then H is said to be *sectorial*. H is *m-sectorial* if it is both m -accretive and sectorial.

If H is sectorial, then the sesquilinear form $h(\cdot, \cdot)$ on $\mathcal{Q}(h) = \mathcal{D}(H)$ defined by

$$h(u, v) = \langle u, Hv \rangle, \quad u, v \in \mathcal{Q}(h)$$

is sectorial and closable (see [23, Theorem VI.1.27]). In particular, any symmetric operator bounded from below defines a closable sectorial form. Conversely, we have (see [23, Theorem VI.2.1, Theorem V.2.6])

Theorem A.2. *Let $h(\cdot, \cdot)$ be a densely defined and closed sectorial form in \mathcal{H} with form domain $\mathcal{Q}(h)$. Then there exists a unique m -sectorial operator H such that $\mathcal{D}(H) \subset \mathcal{Q}(h)$ and*

$$h(u, v) = \langle u, Hv \rangle \text{ for } u \in \mathcal{Q}(h) \text{ and } v \in \mathcal{D}(H).$$

If, in addition, $h(\cdot, \cdot)$ is symmetric and bounded from below, then the associated m -sectorial operator H is self-adjoint with the same lower bounded.

The second part of the above theorem is well-known and widely used in the theory of Schrödinger operator. We also used the following result (see [23, Lemma VI.3.1]).

Theorem A.3. *Let $h(\cdot, \cdot)$ be a densely defined, symmetric, nonnegative closed form with the associated nonnegative self-adjoint operator H . Let $q(\cdot, \cdot)$ be a form relatively bounded with respect to h so that*

$$|q(u, u)| \leq Ch(u, u), \quad u \in \mathcal{Q}(h)$$

for some $C \geq 0$. Then there is $B \in \mathcal{L}(\mathcal{H})$ with $\|B\| \leq \epsilon C$ such that

$$q(u, v) = \langle \sqrt{H}u, B\sqrt{H}v \rangle, \quad u, v \in \mathcal{Q}(h) = \mathcal{D}(\sqrt{H}),$$

where $\epsilon = 1$ or 2 according as q is symmetric or not.

B Justification of (37)

In this section, we justify (37), i.e.,

$$2\Xi_1 c_{z, \lambda_0} \left\| \frac{\lambda + \Xi_1^{-1} \Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} < 1, \quad (58)$$

where $z \in \mathbb{C} \setminus \sigma(H_\Lambda(A, V))$, $\lambda_0 < \min\{0, E_0\}$, $\Xi_1 = \frac{2s}{1-\Theta_1}$, $\Xi_2 = [\frac{2s\Theta_2}{1-\Theta_1} + \frac{1}{2}(\frac{1}{s} + s)]a_0^2$ and $c_{z, \lambda_0} = \left\| \frac{\lambda - \lambda_0}{\lambda - z} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))}$. We here discuss two classes of conditions that guarantee (58). The condition separating them is $\Xi_1^{-1} \Xi_2 = -\lambda_0$.

(i) We note that due to the fact that $\sigma(H_\Lambda(A, V))$ contains a sequence tending to infinity, there holds $\left\| \frac{\lambda + \Xi_1^{-1}\Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} \geq 1$. So the best choice is $\left\| \frac{\lambda + \Xi_1^{-1}\Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} = 1$, which holds if and only if $\Xi_1^{-1}\Xi_2 \leq -\lambda_0$ since $\inf \sigma(H_\Lambda(A, V)) + \Xi_1^{-1}\Xi_2 \geq 0$. By making a_0 small enough, the condition $\Xi_1^{-1}\Xi_2 \leq -\lambda_0$ is readily satisfied. Thus (58) reduces to

$$2\Xi_1 c_{z, \lambda_0} < 1. \quad (59)$$

Note $\lim_{\lambda \rightarrow \infty} \frac{\lambda - \lambda_0}{\lambda - z} = 1$ pointwise in $z \in \rho(H_\Lambda(A, V))$ and $\lambda_0 < \min\{0, E_0\}$, which implies that $c_{z, \lambda_0} \geq 1$. Hence, if (59) holds, then automatically, $\Xi_1 < 1$.

For any fixed $z \in \rho(H_\Lambda(A, V))$ and $\lambda_0 < \min\{0, E_0\}$, there exists s so that (59) is satisfied. Moreover, s can not be chosen to be independent of z or λ_0 because of the facts that

$$\lim_{\substack{z \in \rho(H_\Lambda(A, V)) \\ \text{dist}(z, \sigma(H_\Lambda(A, V))) \rightarrow 0}} c_{z, \lambda_0} = \infty \quad \text{pointwise in } \lambda_0 < \min\{0, E_0\}.$$

or

$$\lim_{\lambda_0 \rightarrow -\infty} c_{z, \lambda_0} = \infty \quad \text{pointwise in } z \in \rho(H_\Lambda(A, V)),$$

respectively.

Explicitly, we can let s be any number satisfying

$$s \in \left(0, \frac{1 - \Theta_1}{4c_{z, \lambda_0}}\right) \quad (60)$$

so that (59) is satisfied, so $\Xi_1 < 1$ holds. Then, by requiring a_0 to satisfy

$$a_0^2 \leq \frac{2s\lambda_0}{\Theta_1 - 1} \left[\frac{2s\Theta_2}{1 - \Theta_1} + \frac{1}{2} \left(\frac{1}{s} + s \right) \right]^{-1}, \quad (61)$$

the condition $\Xi_1^{-1}\Xi_2 \leq -\lambda_0$ holds. Consequently, any pair (s, a) satisfying (60) and (61) guarantees (58).

(ii) Now, we require $\Xi_1^{-1}\Xi_2 \geq -\lambda_0$. Then, $\frac{\lambda + \Xi_1^{-1}\Xi_2}{\lambda - \lambda_0}$, as a function of λ , is decreasing on (λ_0, ∞) , which implies

$$\left\| \frac{\lambda + \Xi_1^{-1}\Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} \leq \frac{\lambda_* + \Xi_1^{-1}\Xi_2}{\lambda_* - \lambda_0} \quad \text{for } \lambda_0 < \lambda_* < \min\{0, E_0\}.$$

We may assume W.L.O.G that $\lambda_* := \frac{4}{5}\lambda_0 < \min\{0, E_0\}$, since, for fixed λ_0 and E_0 , we can always take λ_* to be λ_0 multiplied by some constant so that $\lambda_0 < \lambda_* < \min\{0, E_0\}$. Hence, let $\lambda_* = \frac{4}{5}\lambda_0$, we have

$$\left\| \frac{\lambda + \Xi_1^{-1}\Xi_2}{\lambda - \lambda_0} \right\|_{L^\infty(\sigma(H_\Lambda(A, V)))} \leq \frac{4\lambda_0 + 5\Xi_1^{-1}\Xi_2}{-\lambda_0}.$$

Therefore, $2\Xi_1 c_{z, \lambda_0} \frac{4\lambda_0 + 5\Xi_1^{-1}\Xi_2}{-\lambda_0} < 1$, i.e, $\Xi_2 < \frac{-\lambda_0}{10c_{z, \lambda_0}} - \frac{4}{5}\lambda_0\Xi_1$, will guarantee (58). Considering $\Xi_1^{-1}\Xi_2 \geq -\lambda_0$, we need

$$-\lambda_0\Xi_1 < \frac{-\lambda_0}{10c_{z, \lambda_0}} - \frac{4}{5}\lambda_0\Xi_1,$$

which holds if and only if $2\Xi_1 c_{z,\lambda_0} < 1$, and the later automatically leads to $\Xi_1 < 1$. In conclusion, to insure (58), we require

$$2\Xi_1 c_{z,\lambda_0} < 1 \quad \text{and} \quad -\lambda_0 \Xi_1 \leq \Xi_2 < \frac{-\lambda_0}{10c_{z,\lambda_0}} - \frac{4}{5}\lambda_0 \Xi_1. \quad (62)$$

Explicitly, if $s > 0$ and $a_0 > 0$ satisfy

$$s \in \left(0, \frac{1 - \Theta_1}{4c_{z,\lambda_0}}\right)$$

and

$$\begin{aligned} & \frac{2s\lambda_0}{\Theta_1 - 1} \left[\frac{2s\Theta_2}{1 - \Theta_1} + \frac{1}{2} \left(\frac{1}{s} + s \right) \right]^{-1} \\ & \leq a_0^2 \\ & < \left(\frac{-\lambda_0}{10c_{z,\lambda_0}} - \frac{8s\lambda_0}{5(1 - \Theta_1)} \right) \left[\frac{2s\Theta_2}{1 - \Theta_1} + \frac{1}{2} \left(\frac{1}{s} + s \right) \right]^{-1}, \end{aligned} \quad (63)$$

then (58) holds.

C The Helffer-Sjöstrand Formula

In this section, we define the class of slowly decreasing smooth functions and review the Helffer-Sjöstrand formula (see [22]), which provides an alternative approach to the spectral theory of self-adjoint operators. The material below is taken from [11].

Definition C.1. A function f is said to be in \mathcal{A} , the class of slowly decreasing smooth functions on \mathbb{R} , if $f \in C^\infty(\mathbb{R})$ and there exist $\mu > 0$ and a sequence of constants $c_n \geq 0$, $n \geq 1$ so that

$$|f^{(n)}(u)| \leq c_n \langle u \rangle^{-n-\mu}, \quad \forall u \in \mathbb{R}, \quad \forall n \geq 1,$$

where $\langle u \rangle \equiv \sqrt{1 + |u|^2}$. We define the norms on \mathcal{A} : for $f \in \mathcal{A}$,

$$\|f\|_n = \sum_{r=0}^n \int_{\mathbb{R}} |f^{(r)}(u)| \langle u \rangle^{r-1} dx, \quad n \geq 1.$$

Let $\tau \in C^\infty(\mathbb{R})$ with $\tau(u) = 1$ if $|u| < 1$ and $\tau(u) = 0$ if $|u| > 2$. For $f \in \mathcal{A}$, the smooth (non analytic) extensions $\tilde{f}_n : \mathbb{C} \rightarrow \mathbb{C}$ of f are defined by

$$\tilde{f}_n(z) = \left\{ \frac{1}{r!} \sum_{r=0}^n f^{(r)}(u)(iv)^r \right\} \sigma(u, v), \quad n \geq 1,$$

where $z = u + iv$ and $\sigma(u, v) = \tau\left(\frac{v}{\langle u \rangle}\right)$. Define $\frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} = \frac{1}{2} \left\{ \frac{\partial \tilde{f}_n(z)}{\partial u} + i \frac{\partial \tilde{f}_n(z)}{\partial v} \right\}$. Direct calculation shows

$$\frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} = \frac{1}{2} \left\{ \sum_{r=0}^n \frac{1}{r!} f^{(r)}(u)(iv)^r \right\} (\sigma_u(u, v) + \sigma_v(u, v)) + \frac{1}{2n!} f^{(n+1)}(u)(iv)^n \sigma(u, v).$$

Obviously, $\sigma(u, v) = 0$ if $|v| \geq 2\langle u \rangle$ and both $\sigma_u(u, v) = 0$ and $\sigma_v(u, v) = 0$ if $|v| \leq \langle u \rangle$ or $|v| \geq 2\langle u \rangle$. Thus, by introducing the sets $U = \{(u, v) \in \mathbb{R}^2 | \langle u \rangle < |v| < 2\langle u \rangle\}$ and $V = \{(u, v) \in \mathbb{R}^2 | 0 < |v| < 2\langle u \rangle\}$, we have

$$\left| \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} \right| \leq C \left\{ \sum_{r=0}^n \frac{1}{r!} |f^{(r)}(u)| \frac{|v|^r}{\langle u \rangle} \right\} \chi_U(u, v) + \frac{1}{2n!} |f^{(n+1)}(u)| |v|^n \chi_V(u, v) \quad (64)$$

for some $C > 0$ only depending on τ , where χ_U and χ_V are characteristic functions for U and V , respectively.

Theorem C.2 ([11]). *Let $f \in \mathcal{A}$ and H be a self-adjoint on a separable Hilbert space. Then the integral*

$$\int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} (H - z)^{-1} du dv$$

converges in operator norm and is independent of n and τ . Moreover,

$$\left\| \int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} (H - z)^{-1} du dv \right\| \leq c \|f\|_{n+1}, \quad \forall n \geq 1,$$

where $c > 0$ is a constant independent of f and n .

It should be pointed out the fact that the constant c is independent of n is due to Germinet and Klein [18]. This follows from the fact that $\frac{2^n}{n!} \rightarrow 0$ as $n \rightarrow \infty$.

We then define for $f \in \mathcal{A}$

$$f(H) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\partial \tilde{f}_n(z)}{\partial \bar{z}} (H - z)^{-1} du dv, \quad (65)$$

which is referred to as the Helffer-Sjöstrand formula. By Theorem C.2, $\|f(H)\| \leq c \|f\|_{n+1}$ for all $n \geq 1$.

References

- [1] M. Aizenman, Localization at weak disorder: some elementary bounds. *Rev. Math. Phys.* 6 (1994), no. 5A, 1163-1182.
- [2] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators. *Comm. Pure Appl. Math.* 35 (1982), no. 2, 209-273.
- [3] J. M. Barbaroux, J. M. Combes and P. D. Hislop, Localization near band edges for random Schrödinger operators. *Helv. Phys. Acta* 70 (1997), no. 1-2, 16-43.
- [4] J. Bourgain and C. Kenig, On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.* 161 (2005), no. 2, 389-426.
- [5] K. Broderix, D. Hundertmark, and H. Leschke, Continuity properties of Schrödinger semigroups with magnetic fields. *Rev. Math. Phys.* 12 (2000), no. 2, 181-225.

- [6] K. Chung and Z. Zhao, *From Brownian motion to Schrödinger's equation*. Grundlehren der Mathematischen Wissenschaften, 312. Springer-Verlag, Berlin, 1995.
- [7] J. M. Combes and P. D. Hislop, Localization for some continuous, random Hamiltonians in d -dimensions. *J. Funct. Anal.* 124 (1994), no. 1, 149-180.
- [8] J. M. Combes, P. D. Hislop and F. Klopp, An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. *Duke Math. J.* 140 (2007), no. 3, 469-498.
- [9] J. M. Combes and L. Thomas, Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. *Comm. Math. Phys.* 34 (1973), 251-270.
- [10] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger operators with application to quantum mechanics and global geometry*. Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin, 1987.
- [11] E. B. Davies, *Spectral theory and differential operators*. Cambridge Studies in Advanced Mathematics, 42. Cambridge University Press, Cambridge, 1995.
- [12] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1987.
- [13] A. Figotin and A. Klein, Localization of classical waves. I. Acoustic waves. *Comm. Math. Phys.* 180 (1996), no. 2, 439-482.
- [14] A. Figotin and A. Klein, Localization of classical waves. II. Electromagnetic waves. *Comm. Math. Phys.* 184 (1997), no. 2, 411-441.
- [15] J. Frölich and T. Spencer, Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Comm. Math. Phys.* 88 (1983), no. 2, 151-184.
- [16] G. B. Folland, *Real analysis. Modern techniques and their applications*. Second edition. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999.
- [17] F. Germinet and A. Klein, Bootstrap multiscale analysis and localization in random media. *Comm. Math. Phys.* 222 (2001), no. 2, 415-448.
- [18] F. Germinet and A. Klein, Operator kernel estimates for functions of generalized Schrödinger operators. *Proc. Amer. Math. Soc.* 131 (2003), no. 3, 911-920 (electronic).
- [19] F. Germinet and A. Klein, A characterization of the Anderson metal-insulator transport transition. *Duke Math. J.* 124 (2004), no. 2, 309-350.
- [20] A. Galindo and P. Pascual, *Quantum mechanics. I*. Translated from the Spanish by J. D. García and L. Alvarez-Gaumé. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1990.

- [21] A. Galindo and P. Pascual, *Quantum mechanics. II*. Translated from the Spanish by J. D. García and L. Alvarez-Gaumé. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1991.
- [22] B. Helffer and J. Sjöstrand, *Équation de Schrödinger avec champ magnétique et équation de Harper*. (French) [The Schrödinger equation with magnetic field, and the Harper equation] Schrödinger operators (Sønderborg, 1988), 118C197, Lecture Notes in Phys., 345, Springer, Berlin, 1989.
- [23] T. Kato, *Perturbation theory for linear operators*. Second edition. Grundlehren der Mathematischen Wissenschaften, Band 132. Springer-Verlag, Berlin-New York, 1976.
- [24] W. Kirsch, *Random Schrödinger operators. A course*. Schrödinger operators (Sønderborg, 1988), 264-370, Lecture Notes in Phys., 345, Springer, Berlin, 1989.
- [25] W. Kirsch, *An invitation to random Schrödinger operators*. With an appendix by Frédéric Klopp. Panor. Synthèses, 25, Random Schrödinger operators, 1C119, Soc. Math. France, Paris, 2008.
- [26] Y. Kitagaki, Wegner estimates for some random operators with Anderson-type surface potentials. *Math. Phys. Anal. Geom.* 13 (2010), no. 1, 47-67.
- [27] A. Klein, *Multiscale analysis and localization of random operators*. Random Schrödinger operators, 121-159, Panor. Synthèses, 25, Soc. Math. France, Paris, 2008.
- [28] A. Klein and A. Koines, A general framework for localization of classical waves. I. Inhomogeneous media and defect eigenmodes. *Math. Phys. Anal. Geom.* 4 (2001), no. 2, 97-130.
- [29] A. Klein and A. Koines, A general framework for localization of classical waves. II. Random media. *Math. Phys. Anal. Geom.* 7 (2004), no. 2, 151-185.
- [30] F. Klopp, An asymptotic expansion for the density of states of a random Schrödinger operator with Bernoulli disorder. *Random Oper. Stochastic Equations* 3 (1995), no. 4, 315-331.
- [31] I. M. Lifshitz, Energy spectrum structure and quantum states of disordered condensed systems. *Uspehi Fiz. Nauk* 83 617-663 (Russian); translated as Soviet Physics Uspekhi 7 1965 549-573.
- [32] L. A. Pastur, Spectra of random selfadjoint operators. *Russian Math. Surveys* 28 (1973), no. 1, 1-67.
- [33] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*. Second edition. Academic Press, Inc. , New York, 1980.
- [34] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press, New York-London, 1975.

- [35] B. Simon, Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)* 7 (1982), no. 3, 447-526.
- [36] B. Simon, *Trace ideals and their applications*. Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.
- [37] B. Simon, *Functional integration and quantum physics*. Second edition. AMS Chelsea Publishing, Providence, RI, 2005.
- [38] P. Stollmann, *Caught by disorder. Bound states in random media*. Progress in Mathematical Physics, 20. Birkhäuser Boston, Inc., Boston, MA, 2001.
- [39] A.-S. Sznitman, *Brownian motion, obstacles and random media*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.
- [40] W. Thirring, *Quantum mathematical physics. Atoms, molecules and large systems*. Second edition. Translated from the 1979 and 1980 German originals by Evans M. Harrell II. Springer-Verlag, Berlin, 2002.
- [41] F. Trèves, *Topological vector spaces, distributions and kernels*. Academic Press, New York-London, 1967.
- [42] F. Wegner, Bounds on the density of states in disordered systems. *Z. Phys. B* 44 (1981), no. 1-2, 9-15, 82A42.